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Spectrum of electromagnetic fluctuations in the Casimir effect

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Abstract. The quantum fluctuations of the electromagnetic field between two parallel plates at zero or finite temperature are analysed. The spectrum of the energy-momentum tensor and the two-point correlations of the field are obtained in analytic form through the use of Hertz potentials in the Lorentz gauge. In particular, the general expression for the transition probability of atomic systems is explicitly worked out.

1. Introduction

The Casimir effect [1, 2] is an important manifestation of the vacuum fluctuations predicted by quantum field theory. Some related effects, such as the influence of confining boundaries on atomic or subatomic systems, were analysed some time ago [3], but their importance has increased recently due to the possibility of performing experiments in microcavities [4]. That is the case, for instance, of the inhibition or enhancement of spontaneous emission by the presence of mirrors.

With this fact in mind, we analyse in this paper the structure of the quantum electromagnetic field between two parallel perfectly conducting plates, either at zero or at finite temperature. In particular, we obtain analytic expressions for the spectrum of the electric and magnetic field separately, and for the full energy and stress tensor.

There are several related previous works. For example: the calculation of the energy-momentum tensor between parallel plates [5]; the derivation of the Casimir effect from the Lorentz force acting on the surfaces of the plates [6]; the energy-momentum tensor at finite temperature for scalar and electromagnetic fields [7]; the spectrum of massless scalar fields [8].

In section 2 of this paper we obtain an explicit expression for the electromagnetic propagator in the Lorentz gauge through the use of Hertz potentials†. Although this approach is not unique, we believe that it has several advantages which make it worth describing the formalism in some detail. In particular, the Hertz potentials are well suited for the study of boundary effects [9] with more general geometries, which we will carry on in forthcoming publications. Thus, we calculate the field correlations of the electric and magnetic field, and extend the results in section 3 to the case of finite temperature. As pointed out by Ford [8], some caution is needed in assigning a frequency spectrum to the Casimir effect; one possible solution is to use spectral weight

† For an application of the Hertz potentials to this kind of problem, see [9].

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functions, but for our purposes it is simpler to interpret the spectrum as a distribution [10], with the understanding that in practice it must be weighted with some function related to a specific measurement.

Finally, in section 4, we present a calculation of spontaneous emission within our formalism. This phenomenon is usually studied in the radiation gauge and/or in the dipole approximation. Here, we show explicitly that the use of Hertz potentials also permits the calculation of the transition probabilities in terms of only the space part of the transition current.

2. Zero temperature

We look for a solution of the vacuum Maxwell equations

$$F^{\mu\nu}{}_{,\nu} = 0 \qquad F_{\mu\nu,\rho} + F_{\rho\mu,\nu} + F_{\nu\rho,\mu} = 0 \tag{2.1}$$

subject to the boundary conditions that, in the Lorentz frame in which the conductors are instantaneously at rest, the normal component of the magnetic field and the tangential components of the electric field vanish on the inner surfaces of the conductors. That is,

$$\begin{aligned} &F_{0i}(x) = 0 \qquad F_{ij}(x) = 0 \\ &\text{for } i, j = 2, 3 \qquad \text{at } x = (t, 0, y, z) \text{ and } (t, a, y, z) \end{aligned} \tag{2.2}$$

where a is the separation between the plates. We chose a Lorentz gauge invariant vector potential A_μ, such that

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} \qquad \Box A_\mu = 0 \qquad \partial^\mu A_\mu = 0. \tag{2.3}$$

The geometry of the problem permits the imposing of the additional constraint

$$\partial_2 A^2 + \partial_3 A^3 = 0 \tag{2.4}$$

which implies the existence of the Hertz potentials ϕ, ψ such that

$$A_\mu = (\psi_{,1}, -\psi_{,0}, \phi_{,3}, -\phi_{,2}). \tag{2.5}$$

In terms of these potentials, the electric and magnetic fields are given by

$$\begin{aligned} \boldsymbol{E} &= (-\nabla_\perp^2 \psi, -\phi_{,03} + \psi_{,12}, \phi_{,02} + \psi_{,13}) \\ \boldsymbol{B} &= (-\nabla_\perp^2 \phi, \phi_{,12} + \psi_{,03}, \phi_{,13} - \psi_{,02}) \end{aligned} \tag{2.6}$$

where

$$\nabla_\perp^2 = \partial_2\partial_2 + \partial_3\partial_3. \tag{2.7}$$

The Maxwell equations (2.1) and (2.2) in terms of the potentials ϕ, ψ are equivalent to

$$\Box\phi = 0 \qquad \Box\psi = 0 \tag{2.8}$$

with the boundary conditions

$$\phi = 0 \qquad \psi_{,1} = 0 \tag{2.9}$$

on the inner surfaces of the conductors. Given these conditions, we will call ϕ and ψ Dirichlet and Neumann fields, respectively.

To quantise the Dirichlet and Neumann fields we write the general solution of equations (2.8) as a linear superposition of primitive modes

$$\begin{aligned}\phi_{l,k_\perp}(x) &= N_{\phi_{l,k_\perp}} \exp(\mathrm{i}\boldsymbol{k}_\perp\cdot\boldsymbol{x})\exp(-\mathrm{i}\omega t)\sin\left(\frac{l\pi}{a}x_1\right)\\ \psi_{l,k_\perp}(x) &= N_{\psi_{l,k_\perp}} \exp(\mathrm{i}\boldsymbol{k}_\perp\cdot\boldsymbol{x})\exp(-\mathrm{i}\omega t)\cos\left(\frac{l\pi}{a}x_1\right)\end{aligned} \tag{2.10}$$

where

$$\omega^2 = |\boldsymbol{k}_\perp|^2 + \left(\frac{l\pi}{a}\right)^2 \qquad l\in\mathbb{N} \qquad \boldsymbol{k}_\perp = (0, k_2, k_3)\in\mathbb{R}^3. \tag{2.11}$$

Imposing the normalisation condition

$$\mathrm{i}\int \mathrm{d}^3x\, \phi^*_{l,k_\perp}(x)\overleftrightarrow{\partial}_0\phi_{l',k'_\perp}(x) = \delta(\boldsymbol{k}_\perp - \boldsymbol{k}'_\perp)\delta_{ll'} \tag{2.12}$$

with $\overleftrightarrow{\partial}_0 = (\overrightarrow{\partial}_0 - \overleftarrow{\partial}_0)$, yields the result

$$\begin{aligned}|N_{\phi_{l,k_\perp}}|^2 = |N_{\psi_{l,k_\perp}}|^2 &= [(2\pi)^2\omega a]^{-1} \qquad l\neq 0\\ |N_{\psi_{0,k_\perp}}|^2 &= [2(2\pi)^2\omega a]^{-1}.\end{aligned} \tag{2.13}$$

Thus the Hertz potentials can be quantised as usual:

$$\begin{aligned}\hat{\phi}(x) &= \sum_{l=0}^{\infty}\int \mathrm{d}^2\boldsymbol{k}_\perp[\phi_{l,k_\perp}(x)\hat{a}_l(\boldsymbol{k}_\perp) + \phi^*_{l,k_\perp}(x)\hat{a}^\dagger_l(\boldsymbol{k}_\perp)]\\ \hat{\psi}(x) &= \sum_{l=0}^{\infty}\int \mathrm{d}^2\boldsymbol{k}_\perp[\psi_{l,k_\perp}(x)\hat{b}_l(\boldsymbol{k}_\perp) + \psi^*_{l,k_\perp}(x)\hat{b}^\dagger_l(\boldsymbol{k}_\perp)]\end{aligned} \tag{2.14a}$$

with the commutation relations

$$[\hat{a}_l(\boldsymbol{k}_\perp), \hat{a}^\dagger_{l'}(\boldsymbol{k}'_\perp)] = \delta(\boldsymbol{k}_\perp - \boldsymbol{k}'_\perp)\delta_{ll'} \qquad [\hat{b}_l(\boldsymbol{k}_\perp), \hat{b}^\dagger_{l'}(\boldsymbol{k}'_\perp)] = \delta(\boldsymbol{k}_\perp - \boldsymbol{k}'_\perp)\delta_{ll'} \tag{2.14b}$$

and all other commutators vanishing.

The correlation functions for the Neumann and Dirichlet fields can be written in a form which is usually obtained by the image method. Defining the Dirichlet and Neumann correlation functions as

$$C^-(x,x') \equiv D(x,x') = \langle 0|\hat{\phi}(x)\hat{\phi}(x')|0\rangle \tag{2.15}$$

$$C^+(x,x') \equiv N(x,x') = \langle 0|\hat{\psi}(x)\hat{\psi}(x')|0\rangle \tag{2.16}$$

respectively, and substituting (2.14) in (2.15) and (2.16), one finds directly that

$$\begin{aligned}C^\pm(x,x') = \sum_{l=0}^{\infty}\tfrac{1}{2}\int \mathrm{d}^2\boldsymbol{k}_\perp|N_{l,k_\perp}|\times\exp[\mathrm{i}\boldsymbol{k}_\perp\cdot(\boldsymbol{x}-\boldsymbol{x}') - \mathrm{i}\omega(t-t')]\\ \times\left\{\cos\left[\left(\frac{l\pi}{a}\right)(x_1 - x'_1)\right] \mp \cos\left[\left(\frac{l\pi}{a}\right)(x_1 + x'_1)\right]\right\}\end{aligned} \tag{2.17}$$

and using the formula

$$\sum_{l=-\infty}^{\infty}\delta\left(k_1 - \frac{l\pi}{a}\right) = \frac{a}{\pi}\sum_{l=-\infty}^{\infty}\cos(2k_1 a l) = \frac{a}{\pi}\sum_{l=-\infty}^{\infty}\exp(\mathrm{i}2k_1 a l) \tag{2.18}$$

one obtains

$$C^{\pm}(x, x') = \sum_{l=-\infty}^{\infty} \int \frac{d^3k}{2(2\pi)^2 3\omega} \exp\{i[k_\perp \cdot (x - x') + 2k_1 al - i\omega(t - t')]\}$$
$$\times \{\exp[ik_1(x_1 - x_1')] \mp \exp[-ik_1(x_1 + x_1')]\}$$
$$= F^-(x, x') \mp F^+(x, x') \tag{2.19}$$

where

$$F^{\mp}(x, x') \equiv -\frac{1}{4\pi^2} \sum_{l=-\infty}^{\infty} \frac{1}{(x_1 \mp x_1' - 2al)^2 + (x_2 - x_2')^2 + (x_3 - x_3')^2 - (t - t')^2}. \tag{2.20}$$

Equations (2.20) can also be obtained from the image method taking into account the parity of primitive modes (2.10). Note that $N + D = 2F^-$ and $N - D = 2F^+$.

An advantage of using Hertz potentials is that one can easily take into account the transversality of the electromagnetic field. The procedure consists in writing $\hat{A}_\mu$ in terms of Hertz potentials and introducing creation and annihilation operators for ϕ and ψ only, and not for each component of A_μ. Thus, using (2.5) and (2.14),

$$\hat{A}_\mu(x) = (\partial_1, -\partial_0, 0, 0) \sum_{l=0}^{\infty} \int d^2k_\perp\, N^{\text{em}}_{\psi_{l,k_\perp}} [\psi_{l,k_\perp}(x) \hat{b}_l(k_\perp) + \psi^*_{l,k_\perp}(x) \hat{b}^\dagger_l(k_\perp)]$$
$$+ (0, 0, \partial_3, -\partial_2) \sum_{l=0}^{\infty} \int d^2k_\perp\, N^{\text{em}}_{\phi_{l,k_\perp}} [\phi_{l,k_\perp}(x) \hat{a}_l(k_\perp) + \phi^*_{l,k_\perp}(x) \hat{a}^\dagger_l(k_\perp)] \tag{2.21}$$

with the commutation relations (2.14*b*).

The electromagnetic vacuum state $|0\rangle$, is taken to be the state which satisfies

$$\hat{a}_l(k_\perp)|0\rangle = \hat{b}_l(k_\perp)|0\rangle = 0 \tag{2.22}$$

for all l and $k_\perp$.

The normalisation constants for the electromagnetic field may be obtained by either demanding that each mode of frequency ω makes the correct contribution to the electromagnetic energy:

$$\int_V d^3x\, \tfrac{1}{2}(E_{l,k_\perp} E^*_{l',k'_\perp} + B_{l,k_\perp} B^*_{l',k'_\perp}) = \tfrac{1}{2}\omega\delta(k_\perp - k'_\perp)\delta_{ll'}$$

or using the Klein-Gordon inner product

$$i \int_V d^3x\, A^{\mu *}_{l,k_\perp} \overleftrightarrow{\partial}_0 A_{l',k'_\perp,\mu} = \delta(k_\perp - k'_\perp)\delta_{ll'}.$$

Both conditions above are satisfied if

$$|N^{\text{em}}_{l,k_\perp}|^2 = |k_\perp|^{-2}. \tag{2.23}$$

It is now straightforward to calculate, from the relation (2.5), the correlation function

$$D_{\mu\nu}(x, x') = \langle A_\mu(x) A_\nu(x') \rangle \tag{2.24}$$

of the electromagnetic potential in terms of the Neumann and Dirichlet correlation functions (henceforth the brackets mean vacuum expectation value). We obtain

$$D_{\mu\nu} = \frac{1}{\nabla_\perp^2} \begin{bmatrix} \partial_1\partial_1' N & -\partial_1\partial_0' N & 0 & 0 \\ -\partial_1\partial_0' N & \partial_0\partial_0' N & 0 & 0 \\ 0 & 0 & \partial_3\partial_3' D & -\partial_2\partial_3' D \\ 0 & 0 & \partial_3\partial_2' D & \partial_2\partial_2' D \end{bmatrix}.$$

The correlation functions of the electromagnetic field can also be evaluated using equations (2.6). In general

$$\langle E_i(x) B_j(x')\rangle = \varepsilon_{ijk} \frac{\partial^2}{\partial t\, \partial x^k} N(x, x') \tag{2.25a}$$

$$\langle B_i(x) E_j(x')\rangle = -\varepsilon_{ijk} \frac{\partial^2}{\partial t\, \partial x^k} D(x, x') \tag{2.25b}$$

and

$$\left.\begin{matrix}\langle E_i(x) E_j(x')\rangle \\ \langle B_i(x) B_j(x')\rangle\end{matrix}\right\} = \left(\delta_{ij}\nabla^2 - \frac{\partial^2}{\partial x^i\, \partial x^j}\right) F^-(x, x')$$

$$\pm \begin{bmatrix} \nabla_\perp^2 & -\partial_{12} & -\partial_{13} \\ -\partial_{12} & -\partial_1^2 - \partial_3^2 & -\partial_{23} \\ -\partial_{13} & -\partial_{23} & -\partial_1^2 - \partial_2^2 \end{bmatrix} F^+(x, x') \tag{2.25c}$$

where $\partial_{ab} \equiv \partial^2/\partial x^a\, \partial x^b$.

As for the energy-momentum tensor, it can be written as

$$T_{\mu\nu}(x) = \lim_{x \to x'} T_{\mu\nu}(x, x') \tag{2.26}$$

where [11]

$$T_{\mu\nu}(x, x') = \langle F_{\mu\sigma}(x) F^{\sigma}{}_{\nu}(x') - \tfrac{1}{4}\eta_{\mu\nu} F_{\rho\sigma}(x) F^{\rho\sigma}(x')\rangle \tag{2.27}$$

and it follows that

$$T_{\mu\nu}(x, x') = 2 \frac{\partial^2}{\partial x^\mu\, \partial x^\nu} F^-(x, x'). \tag{2.28}$$

We now apply the above formalism to the calculation of the correlation spectrum of the electromagnetic field as detected in an inertial system at rest with respect to the plates. For this purpose, we chose a fixed point at a distance x from one of the plates and evaluate equations (2.25) in the limit $x \to x'$ and $t - t' = \sigma$. The result is

$$\langle E_i B_j \rangle = 0 \tag{2.29a}$$

$$\tfrac{1}{2}(\langle E_i E_j\rangle + \langle B_i B_j\rangle) = \frac{1}{\pi^2} \sum_{l=-\infty}^{\infty} \frac{[\sigma^2 + (2al)^2]\delta_{ij} - 2(2al)^2 n_i n_j}{[(\sigma - \mathrm{i}\varepsilon)^2 - (2al)^2]^3} \tag{2.29b}$$

$$\tfrac{1}{2}(\langle E_i E_j\rangle - \langle B_i B_j\rangle) = \frac{1}{\pi^2} \sum_{l=-\infty}^{\infty} \frac{-[\sigma^2 + 4(x - al)^2]\delta_{ij} - 2\sigma^2 n_i n_j}{[(\sigma - \mathrm{i}\varepsilon)^2 - 4(x - al)^2]^3} \tag{2.29c}$$

where it is understood that $\langle E_i E_j\rangle = \langle E_i(t+\sigma, \mathbf{x}) E_j(t, \mathbf{x})\rangle$ and similarly for $\langle B_i B_j\rangle$ and $\langle E_i B_j\rangle$.

The Fourier transforms of these expressions are (see the appendix)

$$\left.\begin{matrix}\langle\widetilde{E_iE_j}\rangle\\ \langle\widetilde{B_iB_j}\rangle\end{matrix}\right\} \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\begin{matrix}\langle E_iE_j\rangle\\ \langle B_iB_j\rangle\end{matrix}\right\}\exp(\mathrm{i}\omega\sigma)\,\mathrm{d}\sigma$$

$$= \frac{\omega^3}{8\pi^2}\sum_{l=-\infty}^{l=\infty}\{[P(2a\omega l)\pm P''(2(x-al)\omega)](\delta_{ij}+n_in_j)$$

$$+[P''(2a\omega l)\pm P(2(x-al)\omega)](-\delta_{ij}+n_in_j)\} \tag{2.30}$$

where we have defined the functions

$$P(\xi)=\frac{\sin\xi}{\xi} \tag{2.31a}$$

$$P''(\xi)=2\frac{\sin\xi}{\xi^3}-2\frac{\cos\xi}{\xi^2}-\frac{\sin\xi}{\xi}. \tag{2.31b}$$

Using the formulae (A.2), it follows that

$$\tfrac{1}{2}(\langle\widetilde{E_iE_j}\rangle+\langle\widetilde{B_iB_j}\rangle)$$

$$=\frac{\omega^3}{6}\delta_{ij}+\frac{\omega^2}{8\pi^2 a}\left[f(2a\omega)(\delta_{ij}+n_in_j)\right.$$

$$\left.+\left(\frac{g(2a\omega)}{2(a\omega)^2}-\frac{g'(2a\omega)}{(a\omega)}-f(2a\omega)\right)(-\delta_{ij}+3n_in_j)\right]. \tag{2.32}$$

The first term on the RHS of this equation corresponds to the zero-point energy; the functions f, g and g' are given by (A.2) and have period π/α; notice that there is no dependence on x (the position inside the plates) in this last equation.

Similarly we find, using (A.3), that

$$\tfrac{1}{2}(\langle\widetilde{E_iE_j}\rangle-\langle\widetilde{B_iB_j}\rangle)$$

$$=\frac{\omega^2}{16\pi a}\frac{\sin[(2m-1)(\pi x/a)]}{\sin(\pi x/a)}(-\delta_{ij}+3n_in_j)$$

$$+\frac{1}{64\pi a}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\sin[(2m-1)(\pi x/a)]}{\sin(\pi x/a)}\right)(\delta_{ij}+n_in_j) \tag{2.33}$$

where the ω dependence of this equation is given through the integer m defined as $(m-1)\pi/a\leqslant\omega\leqslant m\pi/a$ (see the appendix). Thus the spectrum of $\langle\widetilde{E_iE_j}\rangle-\langle\widetilde{B_iB_j}\rangle$ is piecewise continuous as a function of the frequency ω. For high frequencies the spectrum oscillates very strongly as a function of x inside the plates. It is worth noticing that the spectrum remains finite at the plates and tends to a constant value as $\omega\to\infty$.

For $\omega<\pi/a$, the above equations reduce to

$$\langle\widetilde{E_iE_j}\rangle=\frac{\omega^2}{4\pi}n_in_j \tag{2.34a}$$

$$\langle\widetilde{B_iB_j}\rangle=\frac{\omega^2}{8\pi}(\delta_{ij}-n_in_j) \tag{2.34b}$$

which reflect the well known fact that the components of the electric modes parallel to the plates and the components of the magnetic modes perpendicular to the plates vanish if their wavelength is larger than $2a$.

We can also integrate (2.32) and (2.33) over all frequencies ω, either by setting $\sigma = 0$ in the regular parts of (2.29) (see [9]), or by integrating each term in the series using formulae (A.4) and (A.5). The final results are

$$\tfrac{1}{2}(\langle E_i E_j\rangle + \langle B_i B_j\rangle) = \frac{1}{6\pi^2}\,\delta_{ij}\int_0^\infty \omega^3\,\mathrm{d}\omega - \frac{\pi^2}{720a^4}(\delta_{ij} - 2n_i n_j) \tag{2.35a}$$

$$\tfrac{1}{2}(\langle E_i E_j\rangle - \langle B_i B_j\rangle) = \frac{\pi^2}{16a^4}\,\frac{[1-\frac{2}{3}\sin^2(\pi x/a)]}{\sin^4(\pi x/a)}\,\delta_{ij}. \tag{2.35b}$$

The first term in (2.35*a*) is the well known zero-point energy and the second term is the contribution of the Casimir effect. Note that the expectation values $\langle E^2(x)\rangle$ and $\langle B^2(x)\rangle$ both diverge as x^{-4} near the bounding plates, but their infinite contributions neatly cancel each other when evaluating the total energy density, thus yielding a finite value; this fact was first noticed by Boyer [12] and discussed at some length by de Witt [13]. On the other hand, the Lorentz invariant $\langle E^2\rangle - \langle B^2\rangle$ does diverge at the plates.

We now calculate the tensor $T_{\mu\nu}(x, x')$ at the event points such that $t - t' = \sigma$ and $\boldsymbol{x} = \boldsymbol{x}'$. The result is

$$T_{\mu\nu}(\sigma) = \pi^{-2}\sum_{l=-\infty}^{\infty}[(\sigma - \mathrm{i}\varepsilon)^2 - (2al)^2]^{-3} \times \begin{bmatrix} 3\sigma^2 + (2al)^2 & 0 & 0 & 0 \\ 0 & \sigma^2 + 3(2al)^2 & 0 & 0 \\ 0 & 0 & \sigma^2 - (2al)^2 & 0 \\ 0 & 0 & 0 & \sigma^2 - (2al)^2 \end{bmatrix}. \tag{2.36}$$

The Fourier transform of this expression is

$$\tilde{T}_{\mu\nu}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(\mathrm{i}\omega\sigma)\,T_{\mu\nu}(\sigma)$$

and the final result after some straightforward algebra is

$$\tilde{T}_{\mu\nu}(\omega) = \frac{1}{6\pi^2}\,\omega^3(\eta_{\mu\nu} + 4t_\mu t_\nu) + \frac{1}{8\pi^2 a^3}\{(2a\omega)^2 f(2a\omega)(-\eta_{\mu\nu} + 4n_\mu n_\nu) + [g(2a\omega) - 2a\omega g'(2a\omega) + (2a\omega)^2 f(2a\omega)](\eta_{\mu\nu} + t_\mu t_\nu - 3n_\mu n_\nu)\} \tag{2.37}$$

where the functions $f(u)$ and $g(u)$ are those given in the appendix, and t_μ is the timelike unit vector defining the rest frame of the plates ($t_\mu t^\mu = -1$).

In order to obtain the total energy-momentum tensor, we must integrate (2.36) directly, using (A.4). The result is

$$T_{\mu\nu} = \frac{1}{6\pi^2}(\eta_{\mu\nu} + 4t_\mu t_\nu)\int_0^\infty \omega^2\,\mathrm{d}\omega + \frac{\pi^2}{720a^4}(\eta_{\mu\nu} - 4n_\mu n_\nu). \tag{2.38}$$

Thus, we recover the usual zero-point plus Casimir energy-momentum tensor.

3. Finite temperature

We turn now to the case of the electromagnetic field between the plates at a finite temperature T. We use the image method to evaluate the corresponding correlation functions, obtaining

$$\left.\begin{matrix} N_\beta \\ D_\beta \end{matrix}\right\} = \frac{1}{4\pi^2} \sum_{\substack{m=-\infty \\ l=-\infty}}^{\infty} \left(\frac{1}{(x_1 - x_1' - 2al)^2 + (x_2 - x_2')^2 + (x_3 - x_3')^2 - (t - t' - \mathrm{i}m\beta)^2} \mp \frac{1}{(x_1 + x_1' - 2al)^2 + (x_2 - x_2')^2 + (x_3 - x_3')^2 - (t - t' - \mathrm{i}m\beta)^2} \right) \quad (3.1)$$

where $\beta = (kT)^{-1}$.

All the formalism developed in section 2 applies in a straightforward way. The field correlation functions are given by (2.25) and the energy-momentum tensor by (2.28) with $F^{\pm}$ substituted by $F_\beta^{\pm}$ ($F_\beta^{\pm} \equiv \frac{1}{2}(N \mp D)$), N by N_β and D by D_β.

As in section 2, we calculate explicitly the field correlations and the energy-momentum tensor for the particular case of an observer at rest between the plates at a distance x from one of them. The results obtained are:

$$\langle \widetilde{E_i B_j} \rangle_\beta = 0 \quad (3.2a)$$

$$\left.\begin{matrix} \langle \widetilde{E_i E_j} \rangle_\beta \\ \langle \widetilde{B_i B_j} \rangle_\beta \end{matrix}\right\} = \left(1 + \frac{2}{\mathrm{e}^{\beta\omega} - 1}\right) \left\{\begin{matrix} \langle E_i E_j \rangle \\ \langle B_i B_j \rangle \end{matrix}\right. \quad (3.2b)$$

$$\tilde{T}^{\beta}_{\mu\nu}(\omega) = \left(1 + \frac{2}{\mathrm{e}^{\beta\omega} - 1}\right) \tilde{T}_{\mu\nu}(\omega) \quad (3.3)$$

again in obvious notation.

We note that in all cases, the Planckian plus the zero-point term are factorised out. Thus, an observer inside a cavity sees a quantum vacuum modified by the presence of the conducting walls, modulated by the hot thermal equilibrium radiation.

The relation between the stress-energy tensor of the EM field and the thermodynamic quantities was discussed by Brown and Maclay [5]. The importance of the free energy has been stressed by several authors [14].

4. Spontaneous emission

Until now, we have studied the behaviour of the quantum electromagnetic field between parallel mirrors. In this section we analyse the coupling of such a field with matter. As mentioned in the introduction, this coupling produces effects which differ from those predicted by quantum electrodynamics in free space. That is the case of the inhibition or enhancement of spontaneous emission in the presence of mirrors [3, 4]. In this section, we work this problem using the mode expansion of the electromagnetic field obtained in section 2. This approach has the advantage of allowing a full multipole expansion in a systematic way, clearly analogous to the usual procedure of quantum electrodynamics in free space.

As usual, let us consider an interaction of matter and radiation given by the term

$$-e \int j^\mu A_\mu \, \mathrm{d}^3x \quad (4.1)$$

where j^μ is the matter current, and the continuity equation

$$\partial^\mu j_\mu = 0 \quad (4.2)$$

guarantees the gauge invariance of physical results. First-order perturbation theory permits the evaluation of the transition probability $W_{\rm fi}^{(l,\boldsymbol{k}_\perp)^{N(D)}}$ per unit time for the emission of a single photon with mode $(l, \boldsymbol{k}_\perp)^N$ or $(l, \boldsymbol{k}_\perp)^{D(*)}$, which turns out to be

$$W_{\rm fi}^{(l,\boldsymbol{k}_\perp)^{N(D)}} = 2\pi |V_{\rm fi}^{(l,\boldsymbol{k}_\perp)^{N(D)}}|^2 \delta(\varepsilon_{\rm i} - \varepsilon_{\rm f} - \omega) \tag{4.3}$$

in the long-time approximation. In this equation $\varepsilon_{\rm i}$ and $\varepsilon_{\rm f}$ are the initial and final energies of the radiating system and $V_{\rm fi}^{(l,\boldsymbol{k})^{N(D)}}$ is the time-independent matrix element of the operator given by (4.1). Explicitly,

$$V_{\rm fi}^{(c,\boldsymbol{k}_\perp)^N} = \left(\frac{1}{(2\pi)^2 2a\omega}\right)^{1/2} \int \mathrm{i}ej_{\rm fi}^{(1)}(\boldsymbol{x}) \exp(\mathrm{i}\boldsymbol{k}_\perp \cdot \boldsymbol{x})\, \mathrm{d}^3\boldsymbol{x} \tag{4.4a}$$

$$V_{\rm fi}^{(l,\boldsymbol{k}_\perp)^N} = \left(\frac{1}{(2\pi)^2 a\omega \boldsymbol{k}_\perp^2}\right)^{1/2} \int \left[ej_{\rm fi}^{(0)}(\boldsymbol{x}) \left(\frac{l\pi}{a} \sin\frac{l\pi}{a} x_1\right) + ej_{\rm fi}^{(1)}(\boldsymbol{x}) \left(-\mathrm{i}\omega \cos\frac{l\pi}{a} x_1\right)\right] \exp(\mathrm{i}\boldsymbol{k}_\perp \cdot \boldsymbol{x})\, \mathrm{d}^3\boldsymbol{x} \qquad l \neq 0 \tag{4.4b}$$

$$V_{\rm fi}^{(l,\boldsymbol{k}_\perp)^D} = \left(\frac{1}{(2\pi)^2 a\omega \boldsymbol{k}_\perp^2}\right)^{1/2} \int \left[ej_{\rm fi}^{(2)}(\boldsymbol{x}) \left(\mathrm{i}k_3 \sin\frac{l\pi}{a} x_1\right) + ej_{\rm fi}^{(3)}(\boldsymbol{x}) \left(\mathrm{i}k_2 \sin\frac{l\pi}{a} x_1\right)\right] \exp(\mathrm{i}\boldsymbol{k}_\perp \cdot \boldsymbol{x})\, \mathrm{d}^3\boldsymbol{x} \tag{4.4c}$$

where

$$j_{\rm fi}^{\mu} \equiv \langle \mathrm{f} | j^\mu | \mathrm{i} \rangle = \exp(\mathrm{i}\omega_0 t) j_{\rm fi}^{\mu}(\boldsymbol{x}) \tag{4.5}$$

and $\omega_0 = \varepsilon_{\rm i} - \varepsilon_{\rm f}$.

Since we are not working in the radiation gauge, the contribution of the time component of the transition current cannot be eliminated. However, the continuity equation (4.2), together with the time dependence (4.5), permits the substitution

$$-j_{\rm fi}^{(0)} \sin\frac{l\pi}{a} \rightarrow \frac{\boldsymbol{k}_\perp \cdot \boldsymbol{j}_{\rm fi}}{\omega_0} \sin\frac{l\pi}{a} x_1 + \frac{(l\pi/a) j_{\rm fi}^{(1)}}{\mathrm{i}\omega_0} \cos\frac{l\pi}{a} x_1 \tag{4.6}$$

in integral (4.4*b*), whenever the radiating system is performing transitions involving bound states. Thus

$$V_{\rm fi}^{(l,\boldsymbol{k}_\perp)^N} = -e\left(\frac{1}{(2\pi)^2 a\omega k_\perp^2}\right)^{1/2} \int \left[\frac{l\pi k_\perp}{a\omega_0} \cdot \boldsymbol{j}_{\rm fi} \sin\frac{l\pi}{a} x_1 + \left(\frac{l^2\pi^2}{\mathrm{i}a^2\omega_0} + \mathrm{i}\omega\right) j_{\rm fi}^{(1)} \cos\frac{l\pi}{a} x_1\right] \times \exp(\mathrm{i}\boldsymbol{k}_\perp \cdot \boldsymbol{x})\, \mathrm{d}^3\boldsymbol{x} \qquad l \neq 0. \tag{4.7}$$

Following the analogous treatments in free space, we can perform a multipole expansion of the electromagnetic waves in expression (4.3), and consider a non-relativistic transition current $j_j : [(e/2\mathrm{i}m)\psi_{\rm f}^* \partial_j \psi_{\rm i}]$. At this stage, some general observations can be inferred from (4.4) and (4.7). Thus, the component of the transition current perpendicular to the plates is coupled just to the Neumann electromagnetic modes. For the parallel modes, both Neumann and Dirichlet modes contribute, but the minimum transition frequency is π/a. This result has already been obtained for dipolar transitions [3] and constitutes the theoretical basis for the calculation of inhibited spontaneous emission [4].

In order to recover the known results, we just have to consider the dipolar approximation in the limiting cases where the dipole moment $\boldsymbol{\mu}_{\rm fi}$ ($\simeq 1/\omega_0$) is either parallel or perpendicular to the plates. In the latter case, the total transition probability $w^{\perp}$ turns out to be

$$w^{\perp}=\omega_0^2|\boldsymbol{\mu}_{fi}|^2\left[\frac{1}{2a}+\frac{1}{2\pi a}\sum_{l=1}^{\infty}\int \mathrm{d}^2\boldsymbol{k}\,\delta(\omega_0-\omega)\left(\frac{k_{\perp}}{\omega}\right)\cos^2\left(\frac{l\pi}{a}x\right)\right]$$
$$=\frac{3\pi A}{\omega_0 a}\left[\frac{1}{2}+\sum_{l=1}^{[\omega_0 a/\pi]}\left(1-\frac{l^2\pi^2}{\omega_0^2a^2}\right)\cos^2\left(\frac{l\pi}{a}x\right)\right] \tag{4.8a}$$

while for the parallel case

$$w^{\parallel}=\frac{1}{2\pi\omega_0 a}\sum_{l=1}^{\infty}\int \mathrm{d}^2\boldsymbol{k}\,\delta(\omega_0-\omega)$$
$$\times\left[\left(\frac{l\pi}{a}\right)^2|\boldsymbol{k}_{\perp}\cdot\boldsymbol{\mu}_{fi}|^2+(k_3\mu_{fi}^{(2)}+k_2\mu_{fi}^{(3)})^2\frac{\omega^2}{\boldsymbol{k}_{\perp}^2}\right]\sin^2\left(\frac{l\pi}{a}x\right)$$
$$=\frac{3\pi A}{2\omega_0 a}\sum_{l=1}^{[\omega_0 a/\pi]}\left(1+\frac{l^2\pi^2}{\omega_0^2a^2}\right)\sin^2\left(\frac{l\pi}{a}x\right) \tag{4.8b}$$

where x stands for the position of the radiating system relative to one of the plates and A corresponds to the Einstein coefficient.

5. Conclusions

In this paper we have systematically used the Hertz potentials to solve the boundary problem of the quantum electromagnetic field between two parallel plates. The use of the Hertz potentials has several advantages: first, the boundary and transversality conditions of the electromagnetic field are easily taken into account; second, it can be applied to more general geometries than the parallel plates [9]; third and last, it provides a relatively simple formalism which permits the calculation of the complete energy-momentum tensor and the correlations of the electromagnetic field. These correlations are essential for the calculation of radiative corrections for confined atomic systems.

The formalism that we have developed applies to any observer. In particular, we have considered a detector at rest between the plates. The spectrum of the energy-momentum tensor, (2.37), and of the correlations, (2.32) and (2.33), turn out to be very complicated, and they are certainly not thermal; in fact, from the mathematical point of view, it has to be interpreted as a distribution (however, a formal relation with a thermal spectrum is still possible, see [15]). The case of an accelerated observer between the plates can also be calculated, and will be presented, together with some applications, in a future publication.

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Appendix

In this appendix we present the main formulae used in the text.

The integrals in (2.30) are

$$\int_{-\infty}^{\infty} \frac{\exp(\mathrm{i}\omega\sigma)\, \mathrm{d}\sigma}{(\sigma - \mathrm{i}\varepsilon)^2 - u^2} = -\frac{2\pi}{u}\sin(\omega u) \tag{A.1a}$$

$$\int_{-\infty}^{\infty} \frac{\exp(\mathrm{i}\omega\sigma)\, \mathrm{d}\sigma}{[(\sigma - \mathrm{i}\varepsilon)^2 - u^2]^2} = \frac{\pi}{u^3}[\sin(\omega u) - \omega u \cos(\omega u)] \tag{A.1b}$$

$$\int_{-\infty}^{\infty} \frac{\exp(\mathrm{i}\omega\sigma)\, \mathrm{d}\sigma}{[(\sigma - \mathrm{i}\varepsilon)^2 - u^2]^3} = \frac{\pi}{4u^5}[(u^2\omega^2 - 3)\sin(\omega u) + 3\omega u \cos(\omega u)]. \tag{A.1c}$$

The infinite sums appearing in the expressions for $\langle E_i E_j \rangle + \langle B_i B_j \rangle$ are of the form [13]

$$\sum_{k=1}^{\infty} \frac{\sin kx}{k} = \frac{\pi - x}{2} = -g''(x) \equiv f(x) \tag{A.2a}$$

$$\sum_{k=1}^{\infty} \frac{\cos kx}{k^2} = \frac{\pi^2}{6} - \frac{\pi x}{2} + \frac{x^2}{4} = g'(x) \tag{A.2b}$$

$$\sum_{k=1}^{\infty} \frac{\sin kx}{k^3} = \frac{\pi^2 x}{6} - \frac{\pi x^2}{4} + \frac{x^3}{12} \equiv g(x) \tag{A.2c}$$

where $g(x + 2\pi) = g(x)$, and $g(x)$ is piecewise $\mathbb{C}^2$.

One of the infinite sums appearing in (2.30) is of the form

$$\sum_{k=-\infty}^{\infty} \frac{\sin(k-\alpha)u}{k-\alpha}$$

which can be written as

$$\frac{\sin(\alpha u)}{\alpha} + 2\cos(\alpha u)\sum_{k=1}^{\infty} \frac{k(-1)^k \sin[k(u+\pi)]}{k^2 - \alpha^2} - 2\sin(\alpha u)\sum_{k=1}^{\infty} \frac{(-1)^k \cos[k(u+\pi)]}{k^2-\alpha^2}.$$

Now, using formulae 1.445 (7 and 8) of Gradshteyn and Ryzhik [13], we finally obtain

$$\sum_{k=-\infty}^{\infty} \frac{\sin(k-\alpha)u}{k-\alpha} = \frac{\pi}{\sin \pi\alpha}\sin[(2m-1)\pi\alpha] \tag{A.3}$$

where m is an integer such that $2(m-1)\pi \leqslant u \leqslant 2m\pi$. Taking the second derivative with respect to α of this last equation, one obtains an expression for the other infinite sum in (2.30).

In order to integrate over frequencies ω, we need the formula

$$\int_0^{\infty} \exp(\mathrm{i}\omega\alpha)\omega^n\, \mathrm{d}\omega = \mathrm{i}^{n+1} n!\, \alpha^{-n-1} + (-\mathrm{i})^n \pi \delta^{(n)}(\alpha)$$

which follows from entry 22, p 360 of [10]. In particular:

$$\int_0^{\infty} \sin(\alpha\omega)\, \mathrm{d}\omega = \alpha^{-1}$$

$$\int_0^{\infty} \omega\cos(\alpha\omega)\, \mathrm{d}\omega = -\alpha^{-2} \tag{A.4}$$

$$\int_0^{\infty} \omega^2 \sin(\alpha\omega)\, \mathrm{d}\omega = -2\alpha^{-3}.$$

Also, by taking derivatives of formula 1.422 (4) in Gradshteyn and Ryzhik [16] one obtains

$$\sum_{k=-\infty}^{\infty} (x-k)^{-4} = n^4 \frac{1-\frac{2}{3}\sin^2(\pi x)}{\sin^4(\pi x)}. \tag{A.5}$$

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